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1985 J. Phys. A: Math. Gen. 18 1337

(http://iopscience.iop.org/0305-4470/18/9/017)

J. Phys. A: Math. Gen. **18** (1985) 1337-1342. Printed in Great Britain

The circular parallel plate capacitor: a numerical solution for the potential

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Received 24 August 1984

Abstract. Numerical values for $V(\rho, z)$ in the vicinity of a parallel plate capacitor have been calculated using both the Love integral-equation method and a relaxation method. The two methods agree. Figures are presented showing equipotentials for three different values of κ, the ratio between plate separation and radius.

1. Introduction

Recently Atkinson *et al* (1983) proposed an analytical solution for the potential of a circular parallel plate capacitor. As Hughes (1984) noted, this solution is incorrect and the reduction of this classic problem to an analytical expression remains unperformed. A correct numerical solution, however, can be obtained. We have solved the problem using two independent numerical methods. Graphs showing equipotentials are presented for plate separation to radius ratios of $\kappa = \frac{1}{3}$, 1 and 3. These graphs, originally derived to aid our own experiments, should be useful to other experimenters and those seeking to derive analytical expressions.

2. Method 1: E R Love

Imagine a circular capacitor of unit radius and of plate separation κ. The capacitor is centred at the origin with its axis coincident with the z axis. The top plate, $z = +\frac{1}{2}\kappa$, $\rho < 1$ is at potential V_0. The bottom plate, $z = -\frac{1}{2}\kappa$, is at potential $-V_0$. Love (1949) found that in this situation the potential at any point in space is given by

$$V(\rho, z) = \frac{V_0}{\pi} \operatorname{Re} \int_{-1}^{1} \left(\frac{1}{[\rho^2 + (z - \frac{1}{2}\kappa + \mathrm{i}t)^2]^{1/2}} - \frac{1}{[\rho^2 + (z + \frac{1}{2}\kappa + \mathrm{i}t)^2]^{1/2}} \right) f(t)\, \mathrm{d}t. \tag{1}$$

Here $f(t)$ is a solution of the integral equation

$$f(x) - \frac{\kappa}{\pi} \int_{-1}^{1} \frac{f(t)\, \mathrm{d}t}{\kappa^2 + (x - t)^2} = 1. \tag{2}$$

This equation for $f(t)$ is of the Fredholm type. It may be solved by the finite-difference method as suggested by Fox and Goodwin (1953). Alternatively, the equation may be

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0305-4470/85/091337+06$02.25

solved iteratively, taking $f(t) = 1$ as the initial guess. This approach was chosen by Love and also by us.

We evaluated $f(t)$ and $V(\rho, z)$ for three different values of κ: $\kappa = \frac{1}{3}$, 1 and 3. To find $f(t)$ the initial guess was substituted into equation (2). The integral was evaluated numerically by breaking the region -1 to $+1$ into 64 intervals. Equation (2) could then be solved for $f(x)$. Although the variables x and t are distinct, the functional dependences $f(x)$ and $f(t)$ are identical. Thus the value of $f(x)$ may be substituted for $f(t)$. The entire process was repeated with the refined value of $f(t)$, yielding a new $f(x)$ and so on. Sufficient iterations were made such that nowhere does the final value for $f(t)$ change from the penultimate value by more than one part in a thousand. In this manner $f(t)$ was determined for 64 values of t. The function $f(t)$ is real, even and positive. It has a maximum value at $t = 0$ and minimum values at $t = \pm 1$. The values of $f(-1)$, $f(0)$ and n, the number of iterations necessary for 0.1% accuracy, are reported in columns 1-4 of table 1 below. Columns 5 and 6 compare our results with those reported by Fox and Goodwin (1953). It is clear from the table that as the plate separation decreases, both $f(0)$ and the required number of iterations increase. This increase occurs because the initial guess $f(t) = 1$ is correct only for a single isolated plate. The closer the plates, the more the charge distribution on one plate affects the other.

Table 1.

				Fox and Goodwin	
κ	n	$f(-1)$	$f(0)$	$f(-1)$	$f(0)$
3	5	1.233	1.256	—	—
1	8	1.638	1.919	1.6364	1.9127
$\frac{1}{3}$	30	2.453	4.014	—	—

Given values of $f(t)$, we used equation (1) to calculate the potential $V(\rho, z)$ numerically. The potential was evaluated at a discrete set of points labelled ρ_i and z_j, spaced by a step interval s. To ensure a reasonably fine grid, the step interval s had the value $\frac{1}{48}$, $\frac{1}{12}$ and $\frac{1}{12}$ for $\kappa = \frac{1}{3}$, 1 and 3 respectively. Because $V(\rho, z)$ is an odd function of z, the potential was evaluated for only one quadrant.

As noted by Love, $f(t)$ may also be used to calculate the capacity. In particular consider the integral

$$I = \int_0^1 f(t)\,\mathrm{d}t = \frac{1}{2}\int_{-1}^{1} f(t)\,\mathrm{d}t. \tag{3}$$

This integral may be interpreted by inspecting equation (1) in the limit that $(\rho^2 + z^2)^{1/2}$ becomes very large. In this limit $V(\rho, z)$ approaches the potential of a dipole and equation (1) reduces to

$$V(\rho, z) = \frac{2V_0 I z \kappa}{\pi(\rho^2 + z^2)^{3/2}}. \tag{4}$$

In the unrationalised units customarily used for this problem the dipole strength is thus $p = 2V_0 I\kappa/\pi$ and the capacity of the two plates is simply I/π. We numerically

integrated $f(t)$ to obtain I. The results of our calculations appear in column 2 of table 2. Column 3 compares our values with previous calculations by Cooke (1958) as summarised by Sneddon (1966).

Table 2.

κ	I (this work)	I (Cooke)
3	1.248	1.2421
1	1.819	1.8208
0.4	—	3.1029
$\frac{1}{3}$	3.522	—

These values of I were used to calculate the capacitance of the system which was then compared to the elementary 'textbook' value $C_{\text{elem}} = A/4\pi d = \frac{1}{4}/\kappa$. We found that $C/C_{\text{elem}} = 1.495$, 2.316 and 4.767 for $\kappa = \frac{1}{3}$, 1 and 3, respectively. It is interesting to note that even for a separation as close as one-third of the radius, the 'textbook' formula underestimates the capacity by 33%.

3. Method 2: relaxation method

In addition to the method outlined above, we used an iterative relaxation method to compute the potential. In this method a computer evaluated $V(\rho_i, z_j) \equiv V(i, j)$ at each point of a 51×51 point array. The points, numbered from $i = 0$ to 50 and $j = -25$ to $+25$, represented spatial locations in the vicinity of the plates.

Initially two small half-capacitor plates were placed at $j = \pm 3$, $0 \leqslant i \leqslant 7$ ($\kappa = 1$) and charged to potential $\pm V_0$. To initialise the field $V(i, j)$ was chosen to vary linearly with j in the region between the plates, and follow the dipole approximation in the region outside the plates. Each plate carried a charge Q determined by the Love method described above. Since the distance to the boundary was large, approximately 8 plate radii, the potential at the boundary could be determined by the dipole approximation to an accuracy of $(\frac{1}{8})^2$ or 1.5%.

We held the potential at the boundary and on the plates fixed and calculated the potential at the remaining field points via a difference equation (5). The computer cycled through the entire field array 400 times changing each $V(i, j)$ until it was related to its four nearest neighbours through the equation

$$V = \frac{1}{4}\Big[V(i, j+1) + V(i, j-1) + \frac{1}{2}\Big(V(i+1, j) + V(i-1, j) + \frac{1}{i}[(i+1)\,V(i+1, j) + (i-1)\,V(i-1, j)]\Big)\Big]. \tag{5}$$

As expected for a problem of cylindrical symmetry, this equation differs from a simple average by giving slightly more weight to the points at larger radius. The equation was chosen to approximate Laplace's equation to terms in third order. The agreement may be verified by doing a Taylor series expansion about $V(i, j)$ on the points $V(i, j+1)$, $V(i+1, j)$, etc, and substituting the resulting terms in equation (5). The first- and third-

order terms vanish by symmetry. The second-order terms vanish when Laplace's equation is enforced, leaving $V = V(i, j)$.

To ensure that the equipotentials would have no cusp on the j axis, we demanded that $V(0, j) = V(1, j)$ after each of the 400 iterations. Sufficient iterations were made to ensure that the change between the final value of each $V(i, j)$ and the penultimate one was less than one part in a thousand.

Once a stable field was calculated the centre of the array was expanded to obtain more detail in the vicinity of the plates. To expand the field new boundaries were chosen at $i = 25$ and $j = \pm 12$. All the points outside the new boundaries were then thrown out. The field within the boundaries was expanded to the original 51×51 size

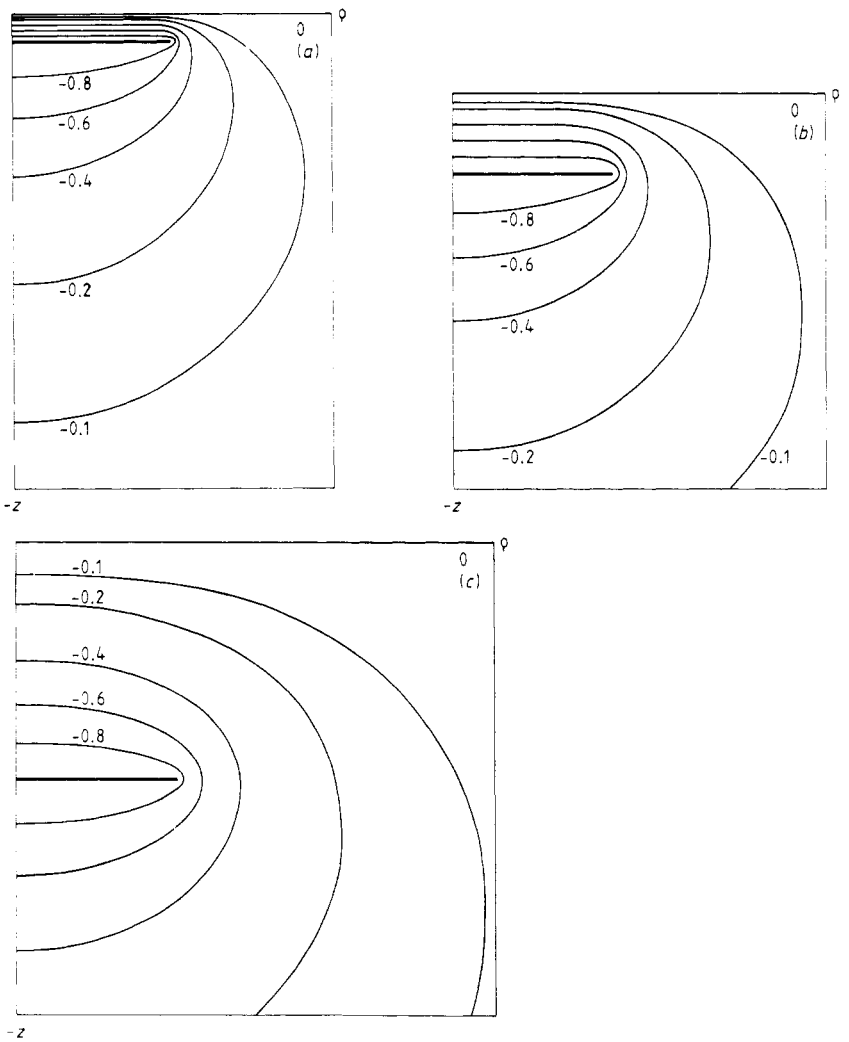

Figure 1. Equipotentials: $V/V_0 = 0$, -0.1, -0.2, -0.4, -0.6, -0.8, -1.0 for $\kappa = \frac{1}{3}$ (*a*), 1 (*b*) and 3 (*c*), respectively. Only one quadrant is shown. The axis of symmetry is vertical, the plate ($V = -V_0$) is bold, the median plane ($V = 0$) is horizontal and at the top of each figure.

by inserting additional field points between the existing points. Holding the potential at the new boundary and plates fixed the difference equation was again employed to generate the potential at each point of the expanded field.

This procedure was repeated twice, yielding plates located at $j = \pm 6$, $0 \leqslant i \leqslant 12$ and their surrounding potential field. The results of this program were compared with those of method 1. The point by point agreement was generally within $\pm 0.01\, V_0$. For a few points at the edge of the plates the relaxation method gave results up to $0.08\, V_0$ higher than the Love method. We believe this discrepancy arises because the capacitor plates used in the relaxation method are not infinitesimally thin.

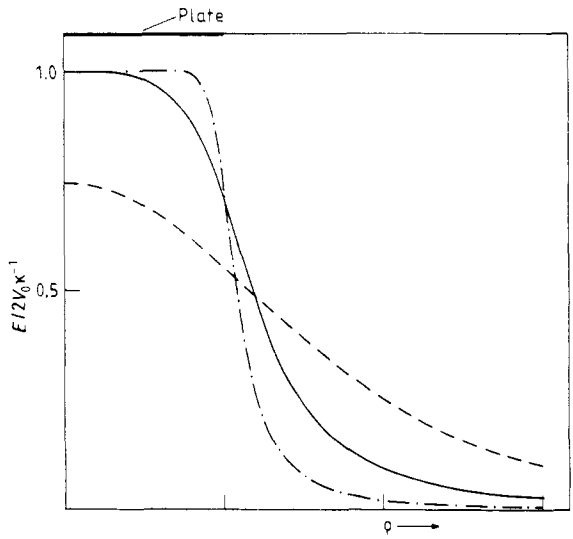

Figure 2. $E/2V_0\kappa^{-1}$ against ρ in the median plane. The plate is the bold line at the top of the figure: —·—, $\kappa = \frac{1}{3}$; ——, $\kappa = 1$; ---, $\kappa = 3$.

4. Results

Plots of the equipotentials generated using Love's method are shown in figure 1. These plots would not be noticeably different if points generated by the relaxation method were used instead. Because $V(\rho, z)$ is an odd function of z, the plots only cover one quadrant. In comparing our plot for $\kappa = 1$ with that of Atkinson *et al* (1983), the reader may note that the equipotentials of the former lack the sharp breaks of the latter. These breaks arose because Atkinson *et al* inadvertantly assumed a charge distribution on the planes $z = \pm\kappa/2$ even for $\rho > 1$.

Finally in figure 2 we plot the electric field in the median plane of the three capacitors as a function of ρ.

Copies of either program are available upon request.

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